$$\rho = \rho_0 e^{-1-x/at},\ v = a(1 + x/at),\ -x_1 \leqslant x < \infty,\ t > 0.$$

It is evident from the above results that in both the similarity solution and in the isothermal case, the velocity of the gas boundary in the cavity is infinite ($\xi_2 = \infty$). However, the total energy remains finite, since at $\xi \to \infty$ the density decreases more rapidly ($g \sim e^{-\xi}$) than the square of velocity increases ($f^2 \sim \xi^2$). The velocity of the shock in both cases is equal to the speed of sound ($\beta = 1$), $\xi_0 = 0.448$ at $\gamma = 1.11$.

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GROUP-INVARIANT SOLUTIONS AND INTERRELATION OF THE PARAMETERS OF THE MOISTURE-TRANSPORT EQUATION

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Study of the moisture-transport processes (during incomplete saturation of a porous medium by water) is a complex and urgent problem. Its component part is the reliable and rapid experimental determination of the parameters of saturated–unsaturated soil. Investigation of the exact solutions of the moisture-transport equation by the method of Lie groups [1] is used for these purposes. A general regularity of the exponential time dependence of the fluid mass flow rate [2], which had been encountered earlier [3], is detected in experiments on dehydration of a soil specimen. Its elucidation in the presence of a strong nonlinearity in the equations is given within the framework of group-invariant solutions. The conditions for expansion of the group result in interactions of the moisture-transport factor, the fundamental hydrophysical dependence, and the moisture, which can turn out to be useful for modeling the moisture transport processes.

1. Formulation of the Problem

The one-dimensional horizontal motion of water in an unsaturated porous medium is described by the moisture-transport equation [4]

$$\theta'_t = [K(p)\, p'_x]'_x, \tag{1.1}$$

where p is the pressure in water column units ($p < 0$ for incomplete saturation); $K(p)$, moisture-transport factor; θ, volume humidity; t, time; and x, longitudinal coordinate.

We introduce a new function which we call the generalized head

$$F(p) = \int K(p)\, dp. \tag{1.2}$$

Then (1.1) takes the form

$$\theta'_t = F''_{x^2}. \tag{1.3}$$

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TABLE 1

Variant	Kind of dependence $\delta = \theta' \mid K$ ($F = \int K dn$)	Value of the coefficient k_1	Interaction of θ and F	Form of the heat-transport factor	Basis of the operators
I	Not determined explicitly	—	—	Not determined explicitly	$X_1 = \frac{\partial}{\partial t}$, $X_2 = \frac{\partial}{\partial x}$, $X_3 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}$
II	$\delta \neq \text{const}$, $\frac{\delta F'}{\delta'} = k_1 F + k_2$	1. $k_1 \neq 0; -1$	$\theta + \theta_0 = [c(k_1 F + k_2)]^{\frac{1+k_1}{k_1}}$	$K = \frac{1}{c(1+k_1)}(\theta + \theta_0)^{-\frac{1}{1+k_1}}\theta'$	$\{X_1, X_2, X_3\}$, $X_4 = t\frac{\partial}{\partial t} + \frac{k_1 F + k_2}{F'}\frac{\partial}{\partial u}$
		2. $k_1 = -1/2$	$F - 2k_2 = \frac{c}{\theta_0 - \theta}$	$K = c(\theta_0 - \theta)^{-2}\theta'$	
		3. $k_1 = 0$	$\theta + \theta_0 = ce^{\frac{1}{k_2}F} k_2$	$K = \frac{k_2}{\theta + \theta_0}\theta'$	
		4. $k_1 = -1$	$\theta + \theta_0 = c\ln(F + k_2)$	$K = \frac{1}{c} e^{\frac{1}{c}(\theta+\theta_0)}\theta'$	
		5. $k_1 = -1/4$	$\theta_0 - \theta = [c(F - 4k_2)]^{-3}$	$K = \frac{1}{3c}(\theta_0 - \theta)^{-4/3}\theta'$	$\{X_1, X_2, X_3\}$, $X_4 = t\frac{\partial}{\partial t} - \frac{1}{4}\frac{F - 4k_2}{F'}\frac{\partial}{\partial u}$, $X_5 = x^2\frac{\partial}{\partial x} + x\frac{F - 4k_2}{F'}\frac{\partial}{\partial u}$
III	$\delta = \text{const}$	—	$\theta + \theta_0 = cF$	$K = \frac{1}{c}\theta'$	$\{X_1, X_2, X_3\}$, $X_4 = \frac{F}{F'}\frac{\partial}{\partial u}$, $X_5 = t\frac{\partial}{\partial t} - \frac{c}{2}x\frac{F}{F'}\frac{\partial}{\partial u}$

As follows from (1.3), the equilibrium distribution of the generalized head is linear: $F = ax + b$, and it will be the limit in the dehydration process.

To close the equation of motion, the fundamental hydrophysical dependence $\theta = \theta(p)$ which is strongly nonlinear in nature, is still necessary. For $p = 0$, total saturation occurs, i.e., the three-phase system (solid skeleton, water, air) becomes a two-phase system (solid skeleton, water) while for $p \to -\infty$, $\theta \to 0$.

In addition to the moisture-transport factor, the diffusivity factor ([4, Sec. 9.10])

$$D = K/(d\theta/dp) \tag{1.4}$$

is often considered in the literature.

Different dependences $D(\theta)$ are constructed on the basis of experimental data. One is the exponential dependence $D = D_0 \exp a\theta$, which is acceptable in the middle part of the θ values, excluding the total saturation neighborhood and the domain of low humidities [4]. Different modifications of the parameters and their possible interdependences (some were proposed empirically) will be obtained purely formally as conditions for the existence of group-invariant solutions.

2. Group Structure of the Moisture-Transport Equation

The group of transformations for the nonlinear heat-conduction equation is calculated in [1, Sec. 21]. Consequently, the infinitesimal operators of this group are naturally equivalent to those obtained below. However, in this paper it turns out to be expedient to trace explicitly the interrelation of the moisture-transport characteristics.

We take the system of moisture-transport equations in the one-dimensional case in the form (for convenience we use the notation $p = u$)

$$\theta'_t = [K(u)u'_x]'_x, \ \theta = \theta(u)$$

and reduce it to one equation

$$\theta'(u)u'_t = [K(u)u'_x]'_x. \tag{2.1}$$

Furthermore, we assume that $\theta'(u) \neq \text{const}$, $K(u) \neq \text{const}$. The number of independent variables is $n = 2$, the number of desired functions is $m = 1$. We use the notation

$$u'_t = p, \ u'_x = q, \ u''_{x^2} = Q, \ u''_{tx} = P.$$

The second continuation of the transformation operator has the form [1]

$$\tilde{\tilde{X}} = \xi^1 \frac{\partial}{\partial t} + \xi^2 \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial u} + \zeta_1 \frac{\partial}{\partial p} + \zeta_2 \frac{\partial}{\partial q} + \zeta_{22} \frac{\partial}{Q}.$$

By standard means we arrive at the following system of governing equations [$\xi^1 = \xi^1(t)$, $\xi^2 = \xi^2(t, x)$, $\eta = \eta(t, x, u)$]:

$$[\eta'_u + (\ln K)'\eta]'_u = 0, \ [\eta'_u + (\ln K)'\eta]'_x = \frac{1}{2}\left(\xi^{2''}_{x^2} - \delta\xi'_t\right),$$

$$\xi^{1'}(t) = 2\xi^{2'}_x + \eta\delta'/\delta, \ \eta''_{x^2} = \delta\eta'_t.$$

Here $\delta(u) = \theta'(u)/K(u)$ and evidently $\delta^{-1} = D$ is given by the relationship (1.4). To expand the group

$$X_1 = \frac{\partial}{\partial t}, \ X_2 = \frac{\partial}{\partial x}, \ X_3 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}$$

it is necessary to require the presence of the dependence

$$\frac{\delta F'}{\delta'} = k_1 F + k_2. \tag{2.2}$$

For $k_1 = -1/4$, the basis of the Lie algebra contains five operators, and if $k_1 \neq -1/4$, then there are four. Integration of (2.2) yields the dependence of $\theta(u)$ on $K(u)$ and the structure of the moisture transport factor. A list of all the variants including $\delta = \text{const}$ is presented in Table 1.

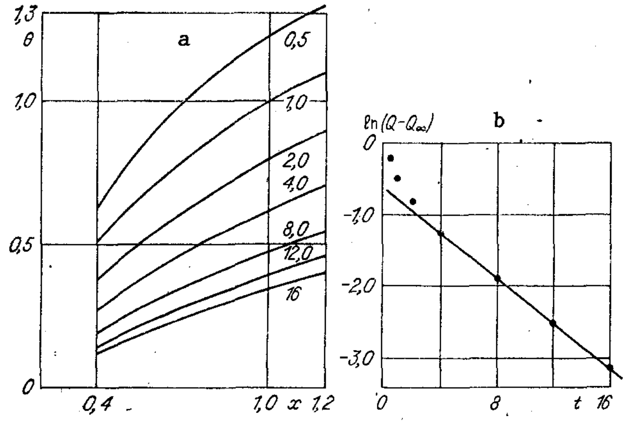

Fig. 1

Considered below are solutions for which the rank of the Lie subalgebra is R = 1 or 2. The number of functionally independent variants will then be $t_0 = n + m - R = 2$ or 1. Giving the invariant manifold of the appropriate group $I_2 = I_2(I_1)$ in the first case and I = const in the second (the number of equations is $\mu = 1$), we see that the necessary existence conditions for an invariant solution $(\delta_0 = m - \mu) \max\{R - n, 0\} \le \delta_0 \le \min\{R - 1, m - 1\}$.

3. Certain Solutions of the Moisture-Transport Equation

A. The variant III (see Table 1) results in the linear equation of motion

$$c\theta'_t = \theta''_{x^2},\ c > 0,$$

the operator

$$X = \alpha X_1 + X_4 = -\alpha\frac{\partial}{\partial t} + \frac{F}{F'}\frac{\partial}{\partial u}$$

has the invariants

$$I_1 = x,\ I_2 = e^{\alpha t}F,$$

which rapidly yields the solution

$$\theta + \theta_0 = e^{-\alpha t}\ f(x), \tag{3.1}$$

while the function f satisfies the equation $f''(x) + c\alpha f(x) = 0$. The Sturm–Liouville problem for it will correspond to a purely exponential dehydration process (3.1), which can be used to determine the moisture-transport factor as a function of the suction pressure if the fundamental hydrophysical dependence is determined in the experiment in parallel [2].

B. Variant II.2 results in an exactly solvable problem since it is converted to a linear parabolic equation.

The equation of moisture motion is the following:

$$\theta'_t = [c(\theta_0 - \theta)^{-2}\theta'_x]'_x. \tag{3.2}$$

In conformity with the method in [5], by considering the solution satisfying the boundary condition $\theta_x'|_{x=0} = 0$, we make the change of variable

$$\xi = \int_0^x g^{-1/2}(t,\ s)\,ds,\ g(t,\ x) \equiv c[\theta_0 - \theta(t,\ x)]^{-2},$$

which will result in

$$\theta'_t = \theta''_{\xi^2} + \left[(\sqrt{g})''_{\theta^2}/(\sqrt{g})'_\theta\right](\theta'_\xi)^2.$$

An example of the exact solution of (3.2) is the moisture distribution

$$\theta = \theta_0 - 1/\sqrt{\exp 2t + x^2},$$

which is evidently of asymptotically exponential nature.

C. A solution of nonexponential type can be obtained in variant II.5 in the operators (R = 2)

$$X_4 = -4t\frac{\partial}{\partial t} + \frac{F - 4k_2}{F'}\frac{\partial}{\partial u},$$

$$\tilde{X}_5 = -\alpha^2 X_2 + X_5 = (x^2 - \alpha^2)\frac{\partial}{\partial x} + x\frac{F - 4k_2}{F'}\frac{\partial}{\partial u}$$

with the general invariant

$$I = t^{1/4}(F - 4k_2)\sqrt{x^2 - \alpha^2}.$$

We hence find the solution

$$\theta = \theta_0 - (3c/4)^{-3/4}\alpha^{3/2}t^{3/4}(x^2 - \alpha^2)^{-3/2},$$

satisfying

$$\theta_t' = \left[(1/3c)(\theta_0 - \theta)^{-4/3}\theta_x'\right]_x'.$$

Therefore, the assertion of the authors of [5] that the equation they examined is the most general class of exact solvability is untrue.

D. And, finally, we turn to the most interesting variant II.4 corresponding to the above-mentioned dependence for D that takes the invariant solution of rank R = 1, constructed with the "classical" operator

$$X_3 = t\frac{\partial}{\partial t} + \frac{x}{2}\frac{\partial}{\partial x}.$$

Setting $F = e^{\theta}$, we see that the solution of the equation resulting from (2.1),

$$\theta_t' = e^{\theta}\left[\theta_{x^2}'' + (\theta_x')^2\right]$$

should be sought in the form $\theta = \theta(t, x) = \theta(\rho)$, $\rho = t/x^2$, which goes over into a system of ordinary differential equations

$$\varphi' = \varphi(e^{\theta}/4\rho^2 - 3/2\rho - \varphi),\ \theta' = \varphi. \tag{3.3}$$

The manifold on which the invariant solution is considered is given by a correct Cauchy problem in the interval $(0, \infty)$ for the system (3.3) under the initial conditions $\theta = 1$, $\varphi = -0.3$, $\rho = 1$. Giving t as a parameter, we can find the moisture distribution in x at any time. The results of a computation are presented in Fig. 1a, where values of t are given. The quantity $\ln(Q - Q_\infty)$ is constructed as a function of time in Fig. 1b, where $Q = \int_{0,4}^{1,2}\theta dx$, $Q_\infty = 0.235$.

The asymptotically exponential nature of the process is evident. The curves in Fig. 1a, b correspond entirely to experimental representations.

It is seen from the figure that the deviation from an exponential law drops rapidly with time. Indeed, if the solution of the equation

$$\theta_t' = (e^{\theta})_{x^2}''$$

is written in the form

$$\theta = \theta_0 + e^{-\alpha t}f_1(x) + e^{-2\alpha t}f_2(x) + \ldots$$

and the exponential is expanded in a series, then we easily obtain

$$-\alpha e^{-\alpha t}f_1 - 2\alpha e^{-2\alpha t}f_2 - \ldots = e^{\theta_0}e^{-\alpha t}\left\{f_1'' + e^{-\alpha t}\left[f_2'' + \frac{1}{2}(f_1^2)''\right] + O(e^{-2\alpha t})\right\}.$$

Now, let $f_1(x)$ satisfy the equation

$$e^{\theta_0} f_1''(x) = -\alpha f_1(x),$$

then we have to determine the next approximation $f_2(x)$

$$-2\alpha f_2(x) = \left[f_2''(x) + \frac{1}{2}(f_1^2)''\right] e^{\theta_0}.$$

These relationships result in the asymptotic

$$\theta = \theta_0 + e^{-\alpha t}\left[f_1(x) + O(e^{-\alpha t})\right].$$

Thus, a group classification of the solutions of the moisture-transport equations in the one-dimensional case permits finding the interrelation between the moisture-transport factor and the moisture, and the fundamental hydrophysical dependence generating the invariant solutions. These solutions result in an exponential mass flow rate of water in time, which corresponds to experiment. The fundamental hydrophysical dependence, which undoubtedly changes from soil to soil, does not affect the mentioned interrelations.

The results obtained can be used as an indicator of the direction for further experimental and theoretical study to find a closed phenomenological description of moisture-transport processes. The investigation of multidimensional solutions, in particular with gravity taken into account, is urgent. In the one-dimensional case of a vertical flow, the necessary conditions for group expansion for $\delta \neq$ const are exactly as in Table 1.

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